Section 3: One-to-one, Onto, and Inverse Functions

- In this section, we will look at three special classes of functions and see how their properties lead us to the theory of counting.
- So far, we have the general notion of a function f:X $\rightarrow$ Y, but in terms of the comparative sizes of the three sets involved (X, Y and f), all we can say is that $|f| = |X|$.
- In this section, we compare |X| with |Y|.

One-to-one Functions

- Definition: A *one-to-one (injective)* function f from set X to set Y is a function such that each x in X is related to a *different* y in Y.

- More formally, we can restate this definition as either: $f:\mathrm{X} \to \mathrm{Y}$ is 1-1 provided

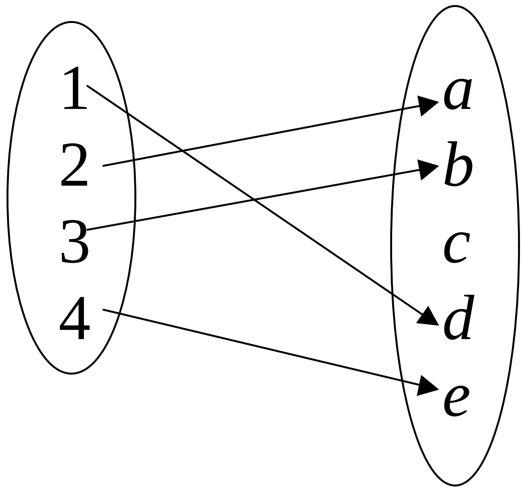
  $$f(x_1) = f(x_2) \text{ implies } x_1 = x_2,$$

  or $f:\mathrm{X} \to \mathrm{Y}$ is 1-1 provided

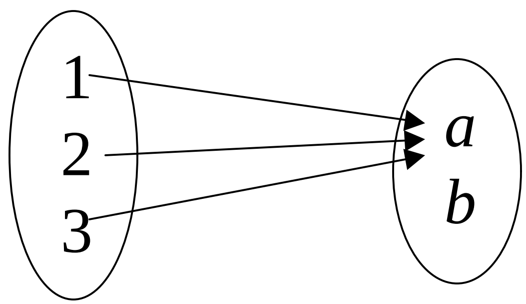
  $$x_1 \neq x_2 \text{ implies } f(x_1) \neq f(x_2).$$

Illustrative Examples

- The function below is 1-1:



This function is not:



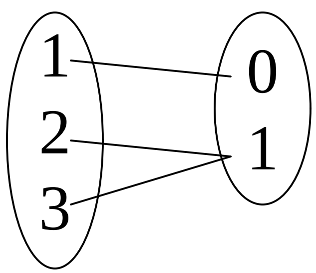
Proving Functions Are 1-1

- If f:**R** $\rightarrow$ **R** is given by $f(x) = 3x + 7$, prove it is one-to-one.
- Proof: Assume $f(a) = f(b)$. Show $a = b$.
  Now $f(a) = f(b)$ means $3a + 7 = 3b + 7$, so $3a = 3b$, therefore $a = b$.

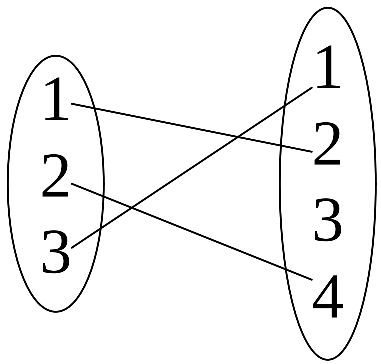
- Why is f:**R** $\rightarrow$ **R** given by $f(x) = x^2$ not 1-1?
- Since $9 = f(3) = f(-3)$, but $3 \neq -3$, the definition is violated.

Onto Functions

- <u>Definition:</u> A function f:X $\rightarrow$ Yis said to be *onto (surjective)* if for every y in Y, there is an x in X such that $f(x) = y$.
- This can be restated as: A function is onto when its image equals its range, i.e. f(X) = Y.
- Examples:



ONTO



NOT ONTO

Testing Onto For Infinite Functions

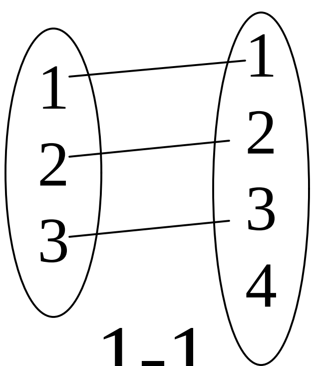
- Show that f:**R** $\rightarrow$ **R** given by $f(x) = 5x - 7$ is onto.
- Let y be in **R**. Then $(y + 7)$ and $(y + 7)/5$ are also real numbers.
  Now $f(\ (y + 7)/5\ ) = 5[(y + 7)/5] - 7 = y$, hence if y is in **R**, there exists an x in **R** such that f(x) = y.
- Is f:**R** $\rightarrow$ **R** given by f(x) = 1/x onto?
- No! There is no x in **R** that has output = 0.

One-to-one Correspondences

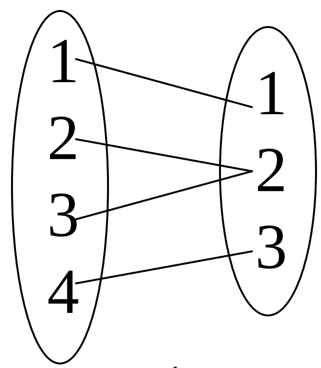
- Definition: A function is called a *one-to-one correspondence (bijection)* if it is one-to-one and onto.
- One-to-one correspondences define the theory of counting. Why?
- If f:X $\rightarrow$ Y is one-to-one, then $|X| \leq |Y|$, and if f is onto, then $|X| \geq |Y|$, so if f is both, $|X| = |Y|$.
- Hence, to count the elements of an unknown set, we create a 1-1 correspondence between the set and a set of known size. Simple!

Inverse Functions

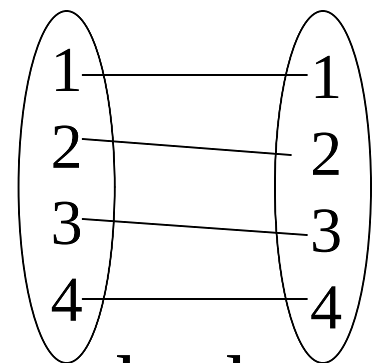
- Recall that the inverse relation is created by inverting all the ordered pairs that comprise the original relation.
- When is the inverse of a function itself a function?



1-1
not onto
(f^{-1} not def.)



onto
not 1-1
(f^{-1} not well-def.)



both
(f^{-1} is a function)

Finding Inverse Functions

- Theorem: If f:X $\rightarrow$ Y is a one-to-one and onto, then f^{-1} is a one-to-one and onto function.
- Given f , how do we find f^{-1}?
- Let f:R $\rightarrow$ R be given by $f(x) = 4x - 1 = y$. Now, swap x and y and solve for y:

$$4y - 1 = x$$
$$4y = x + 1$$
$$y = \frac{x + 1}{4}.$$

- Thus $f^{-1}(x) = (x + 1)/4$.